ON THE VARIOUS FORMS OF THE CONSERVATION EQUATIONS IN TWO-PHASE FLOW

G. Yadigaroglu and R. T. Lahey, Jr.

University of California, Berkeley, CA 94720, U.S.A. and Rensselaer Polytechnic Institute, Troy, NY 12181, U.S.A.

Abstract—The conservation equations for one-dimensional two-phase flow are derived from first principles. The effects of the radial distributions of velocities, enthalpies, and void fraction are taken into account through the use of correlation coefficients. Several simplified separated-flow model formulations that have appeared in the literature are derived from these equations by specializing the values of the correlation coefficients. The equivalence of these formulations under certain assumptions is demonstrated. Finally, new Lagrangian forms of the conservation equations, written in terms of the velocities of the center of mass, momentum, and energy are presented.

1. INTRODUCTION

A vast amount of two-phase flow thermo-hydrodynamics work has been done using various forms of the mass, momentum, and energy conservation equations. Only relatively recently, however, has a rigorous and systematic study of the correct forms of the two-phase conservation equations been undertaken (Delhaye 1968; Ishii 1975). These studies have shown that, although it is possible to write exact conservation equations, the degree of complexity of such forms, and the large amount of detailed local and statistical information required for their solution, prohibit their use in practical applications. Thus, it is necessary to use simplified forms of the conservation equations for many problems of practical significance.

Most of the previous simplified formulations of the conservation equations have either ignored or incompletely taken into account the radial distributions of the phases and of the phase velocities and enthalpies. These radial distribution effects are explicitly taken into account in this paper through the use of correlation coefficients. The paper also reviews a number of simplified one-dimensional formulations that have appeared in the literature and discusses their relation to the more rigorous formulations that are derived here. The equivalence of the various formulations under certain assumptions is shown. Finally, new alternative forms of the conservation equations that might be useful for certain classes of problems are presented.

All turbulent flows have strong random components. Two-phase flows exhibit an even higher degree of randomness due to the local statistical variations of the void fraction, which can be defined as the probability that a point in space will be surrounded by the gaseous phase at some time "t" (Delhaye 1968). The link between this theoretical definition of the local, instantaneous void fraction and a typical experimental measurement is provided by the ergodic hypothesis (Delhaye 1968), which essentially states that, for a stationary process, the average of the simultaneous observations of N experiments is equivalent to the average of N observations in time of an identical experiment. There are, however, other possible definitions of the local instantaneous void fraction. For instance, one can consider a small volume element containing both phases in variable proportions and in this manner arrive at a local, instantaneous volumetric concentration. In this approach the two-phase mixture is considered similar to an emulsion. The work of Wundt (1967) and Bijwaard *et al.* (1964) is based on this approach.

The two-phase flow equations can be rigorously written in terms of local, instantaneous, random variables (Delhaye 1974; Ishii 1975). Since the use of one-dimensional time-dependent equations is often imposed by practical considerations, the problem of the correct statistical and cross-sectional averaging in a flow passage arises. For the purposes of this paper the local void fraction α, as well as the other flow variables, are statistically-averaged deterministic quantities.

The validity of the resulting equations presented here has been previously discussed by Vernier & Delhaye (1968) and Kocamustafaogullari (1971).

Adopting commonly used notation (e.g. Zuber & Findlay 1965), we indicate by angle brackets $(\langle\rangle)$ quantities averaged over the entire cross-sectional area, A_{x-s},

$$\langle f \rangle \triangleq \frac{1}{A_{x-s}} \iint_{A_{x-s}} f \, dA, \tag{1}$$

while for quantities clearly related to the gaseous or liquid phase only,

$$\langle f_L \rangle_L \triangleq \frac{1}{A_{x-s}\langle 1-\alpha\rangle} \iint_{A_{x-s}} f_L(1-\alpha) \, dA = \frac{\langle (1-\alpha)f_L\rangle}{\langle 1-\alpha\rangle},$$

$$\langle f_G \rangle_G \triangleq \frac{1}{A_{x-s}\langle\alpha\rangle} \iint_{A_{x-s}} f_G\alpha \, dA = \frac{\langle\alpha f_G\rangle}{\langle\alpha\rangle}, \tag{2}$$

where the subscripts G and L denote the gas and liquid phases, respectively. Thus, for example, the average velocities of the two phases are defined as,

$$\langle u_L \rangle_L \triangleq \frac{1}{A_{x-s}\langle 1-\alpha\rangle} \iint_{A_{x-s}} u_L(1-\alpha) \, dA = \frac{\langle u_L(1-\alpha)\rangle}{\langle 1-\alpha\rangle},$$

$$\langle u_G \rangle_G \triangleq \frac{1}{A_{x-s}\langle\alpha\rangle} \iint_{A_{x-s}} u_G\alpha \, dA = \frac{\langle u_G\alpha\rangle}{\langle\alpha\rangle},$$

and the average enthalpies as,

$$\langle h_L \rangle_L \triangleq \frac{1}{A_{x-s}\langle 1-\alpha\rangle} \iint_{A_{x-s}} h_L(1-\alpha) \, dA = \frac{\langle h_L(1-\alpha)\rangle}{\langle 1-\alpha\rangle},$$

$$\langle h_G \rangle_G \triangleq \frac{1}{A_{x-s}\langle\alpha\rangle} \iint_{A_{x-s}} h_G\alpha \, dA = \frac{\langle h_G\alpha\rangle}{\langle\alpha\rangle}.$$

2. FLOW MODELS

A number of hypotheses must be made about the structure of two-phase flow in order to reduce the conservation equations into tractable forms. The simplest possible hypothesis, namely that the two phases have equal velocities and temperatures everywhere in the flow channel, results in the so-called homogeneous, equilibrium model. The homogeneous, equilibrium flow equations can be derived from the more general "separated-flow" model equations by arbitrarily equating the velocities and temperatures of the two phases. As pointed out by Bouré (1975), this technique clearly shows the restrictions implied on the evolution of the two-phase mixture. The obvious advantage of the homogeneous, equilibrium model equations is that they have a form similar to that of the single-phase conservation equations and are often analytically tractable. These equations have been dealt with extensively in the literature and thus will not be considered further here.

The classical "separated" or "slip" flow model equations can be derived by assuming that the products of certain cross-sectional averages of the various quantities appearing in the conservation equations are equal to the product of the averages.† This is essentially equivalent to assuming flat velocity profiles for both phases, although the assumption of flat void fraction distributions is also sufficient in some cases.

†For a rigorous study of the approximations involved see Vernier & Delhaye (1968).

The two-phase conservation equations are first derived in this paper for arbitrary void fraction and velocity profiles. To accomplish this, a number of cross-sectional-average products are separated into products of averages through the use of correlation coefficients. The classical separated-flow equations are then obtained by specializing the values of these coefficients. These equations are subsequently transformed into several equivalent forms that have appeared in the literature.

In diabatic two-phase flow, the radial variations of the enthalpies h_G and h_L may be significant and can be conveniently accounted for by appropriate correlation coefficients. Since the phase enthalpies are unique functions of phase pressure and density one should also consider the radial variation of these parameters. However, for most problems of practical concern, the radial variation of the phase pressures p_G and p_L and densities ρ_G and ρ_L can be neglected, and thus the cross-sectional phase-average pressure and density are used throughout this paper. This greatly simplifies the subsequent analysis and minimizes the number of required correlation coefficients.

The conservation equations are first derived for an idealized annular flow in a one-dimensional duct of slowly varying cross-sectional area (Lahey 1974). This is done only in order to better show the physical significance of the various terms, since the resultant conservation equations are of more general applicability. Indeed Meyer (1960) has derived an equivalent set of equations starting from general integral expressions and Ishii (1975) has derived generalized conservation equations which reduce to those in this paper. Moreover, for transient phenomena of practical concern, the equations of this paper can be obtained rigorously from the exact formulations of Delhaye (1968), Vernier & Delhaye (1968) and Kocamustafaogullari (1971).

Although the phase conservation equations are derived first in this paper, these are subsequently combined to yield the mixture conservation equations. Réocreux (1973) and Bouré & Réocreux (1972) have noted that when this is done, restrictive assumptions are implicitly made and thus the conservation equation of the mixture is less general. For example, it may be necessary to assume equal phase pressures, saturation of one or both phases, a certain slip law, etc. Réocreux has applied the complete system of the six conservation equations to two-phase critical flow problems. Although such a system of equations is general and exact, the phase-to-phase and phase-to-wall interaction terms must be known before a solution is possible. Unfortunately, accurate specification of these interaction terms is currently beyond the state-of-the-art. Thus, in this paper the various forms of the two-phase mixture conservation equations are discussed, since these, coupled with various empirical wall interaction and void models, are sufficient for a large number of practical problems.

3. THE MASS CONTINUITY EQUATIONS

With reference to figure 1, consider the mass conservation equations, written separately for each phase (Lahey 1974; Zuber 1967a),

$$\frac{\partial}{\partial t}(\rho_G\langle\alpha\rangle A_{x-s}) + \frac{\partial}{\partial z}(\rho_G\langle\alpha\rangle\langle u_G\rangle_G A_{x-s}) = \delta w', \qquad [3]$$

$$\frac{\partial}{\partial t}(\rho_L\langle 1-\alpha\rangle A_{x-s}) + \frac{\partial}{\partial z}(\rho_L\langle 1-\alpha\rangle\langle u_L\rangle_L A_{x-s}) = -\delta w', \qquad [4]$$

where $\delta w'$ is the amount of liquid evaporated per unit axial length in the differential control volume. In the most general case $\delta w'$ must be obtained from an appropriate vaporization model. For the special case of bulk boiling, where thermodynamic equilibrium is assumed, and flashing is negligible, a simple heat balance yields the following approximation,
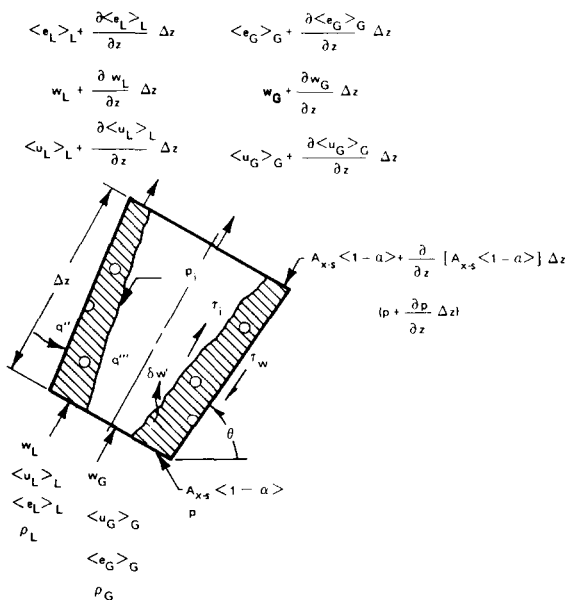
$$\delta w' = q'' P_h / h_{LG} = q' / h_{LG}, \qquad [5]$$

Figure 1. Control volume for idealized annular flow.

in which q'' denotes the heat flux, q' denotes the heat addition per unit length and h_{LG} is the latent heat of vaporization. P_h denotes the heated perimeter.

Equations [3] and [4] can be added to obtain the continuity equation for the two-phase mixture,

$$\frac{\partial}{\partial t}(\langle\rho\rangle A_{x-s}) + \frac{\partial}{\partial z}(GA_{x-s}) = 0 , \qquad [6]$$

where $\langle\rho\rangle = \langle 1 - \alpha\rangle\rho_L + \langle\alpha\rangle\rho_G$ and the mass flux, G, is given by,

$$G = \rho_L\langle 1 - \alpha\rangle\langle u_L\rangle_L + \rho_G\langle\alpha\rangle\langle u_G\rangle_G . \qquad [7]$$

Zuber & Dougherty (1967) and Zuber (1967b) have noted that two separate phase continuity equations are needed, since the two-phase mixture continuity equation alone normally cannot correctly describe the propagation phenomena taking place in boiling channels. Moreover, Zuber & Staub (1966, 1967) have shown that the so-called void propagation equation, together with the two-phase continuity equation of the mixture, form a set equivalent to the two separate phase continuity equations.

4. THE MOMENTUM EQUATIONS

The momentum equations for each phase are examined next. Considering the control volume of figure 1, conservation of momentum for the liquid phase implies,

$$-\frac{\partial}{\partial z}(p_L\langle 1 - \alpha\rangle A_{x-s})\Delta z + p_i\frac{\partial(\langle 1 - \alpha\rangle A_{x-s})}{\partial z}\Delta z - g\rho_L\langle 1 - \alpha\rangle A_{x-s}\Delta z \sin\theta - \tau_{w_L}P_{f_L}\Delta z + \tau_i P_i\Delta z$$

$$= \frac{\partial}{\partial t}(\rho_L\langle 1 - \alpha\rangle A_{x-s}\langle u_L\rangle_L\Delta z) + \frac{\partial}{\partial z}[\rho_L A_{x-s}\langle(1 - \alpha)u_L^2\rangle]\Delta z + \delta w' U_i\Delta z . \qquad [8]$$

The first term in this equation originates from the external forces applied at the ends of the control volume. The second term is the projection on the z axis of the liquid flow area variation (due both to duct and to liquid-phase flow area changes) times the interfacial pressure p_i. The

symbols τ_i and τ_{w_L} denote the shear stresses at the liquid–vapor interface and at the wall, respectively. P_f and P_i denote the wetted and interface perimeters, respectively.

The first two terms on the R.H.S. of [8] are the temporal and spatial acceleration terms, respectively. The last term represents the momentum exchange due to mass transfer between the phases, and deserves special attention.

The appropriate velocity for momentum exchange due to mass transfer is taken to be the interfacial velocity U_i. It is assumed that this term can be written as,

$$U_i \stackrel{\Delta}{=} \eta \langle u_L \rangle_L + (1 - \eta)\langle u_G \rangle_G .$$

The momentum-transfer parameter η has been introduced in order to maintain generality. The classical assumption is to take $\eta = 1$, and thus assume that the average liquid-phase velocity alone is important in evaporative momentum transfer. Alternatively, based on entropy production considerations, Wallis (1969) recommends $\eta = 1/2$. This implies that the effective velocity of the evaporated liquid is $(\langle u_L \rangle_L + \langle u_G \rangle_G)/2$. Since evaporation is assumed to occur at the interface this choice does not appear to be unreasonable and leads to momentum equations for each phase that are symmetrical for evaporation and condensation. Nevertheless, for the flat velocity profiles typical of highly turbulent flows, a value of η close to unity is normally a better choice.

The first two terms on the L.H.S. of [8] can be expanded, and assuming that $p_i \cong p_L$, can be simplified to yield to first order,

$$-\langle 1 - \alpha \rangle A_{x-s} \frac{\partial p_L}{\partial z} \Delta z .$$

This is an interesting result, since the same expression is obtained by neglecting axial flow area changes of the duct and liquid-phase area changes due to evaporation. Thus, for a variable-area duct the liquid-phase momentum equations becomes

$$-\langle 1 - \alpha \rangle \frac{\partial p_L}{\partial z} - g \rho_L \langle 1 - \alpha \rangle \sin \theta - \frac{\tau_{w_L} P_{f_L}}{A_{x-s}} + \frac{\tau_i P_i}{A_{x-s}} = \frac{\partial}{\partial t}(\rho_L \langle 1 - \alpha \rangle \langle u_L \rangle_L)$$

$$+ \frac{1}{A_{x-s}} \frac{\partial}{\partial z}(\rho_L A_{x-s} \langle (1 - \alpha)u_L{}^2 \rangle) + \frac{\delta w'}{A_{x-s}}(\eta \langle u_L \rangle_L + (1 - \eta)\langle u_G \rangle_G). \qquad [9]$$

The momentum equation for the vapor phase can be derived in a similar manner:

$$-\langle \alpha \rangle \frac{\partial p_G}{\partial z} - g \rho_G \langle \alpha \rangle \sin \theta - \frac{\tau_i P_i}{A_{x-s}} - \frac{\tau_{w_G} P_{f_G}}{A_{x-s}} = \frac{\partial}{\partial t}(\rho_G \langle \alpha \rangle \langle u_G \rangle_G)$$

$$+ \frac{1}{A_{x-s}} \frac{\partial}{\partial z}(\rho_G A_{x-s} \langle \alpha u_G{}^2 \rangle) - \frac{\delta w'}{A_{x-s}}(\eta \langle u_L \rangle_L + (1 - \eta)\langle u_G \rangle_G). \qquad [10]$$

Mixture momentum equations

Using the definition, $P_f \tau_w \stackrel{\Delta}{=} P_{f_L} \tau_{w_L} + P_{f_G} \tau_{w_G}$, and assuming that the liquid and vapor phase static pressures, p_L and p_G are equal to the static pressure of the system, p, [9] and [10] can be added to yield the momentum equation of the two-phase mixture,

$$-\frac{\partial p}{\partial z} - g \langle \rho \rangle \sin \theta - \frac{\tau_w P_f}{A_{x-s}} = \frac{\partial}{\partial t}(\rho_L \langle 1 - \alpha \rangle \langle u_L \rangle_L + \rho_G \langle \alpha \rangle \langle u_G \rangle_G)$$

$$+ \frac{1}{A_{x-s}} \frac{\partial}{\partial z}(\rho_L A_{x-s} \langle (1 - \alpha)u_L{}^2 \rangle + \rho_G A_{x-s} \langle \alpha u_G{}^2 \rangle). \qquad [11]$$

At this point it is necessary to separate the products that appear inside the angle brackets of the spatial acceleration term, in order to put [11] into a directly usable form. Following the general approach of Hancox & Nicoll (1971), we define the correlation coefficients,

$$C_L \overset{\Delta}{=} \frac{\langle(1-\alpha)u_L^2\rangle}{\langle 1-\alpha\rangle\langle u_L\rangle_L^2},$$

$$C_G \overset{\Delta}{=} \frac{\langle\alpha u_G^2\rangle}{\langle\alpha\rangle\langle u_G\rangle_G^2}, \qquad\qquad [12]$$

and introduce these into [11]. Using [7] and the following expressions for the flow quality,

$$\langle x\rangle = \frac{\langle u_G\rangle_G \rho_G \langle\alpha\rangle}{G},$$

$$1-\langle x\rangle = \langle 1-x\rangle = \frac{\langle u_L\rangle_L \rho_L \langle 1-\alpha\rangle}{G}, \qquad\qquad [13]$$

[11] can be transformed into,

$$-\frac{\partial p}{\partial z} - g\langle\rho\rangle \sin\theta - \frac{\tau_w P_f}{A_{x-s}} = \frac{\partial G}{\partial t} + \frac{1}{A_{x-s}}\frac{\partial}{\partial z}\left(\frac{G^2 A_{x-s}}{\rho'}\right), \qquad\qquad [14]$$

where,

$$\frac{1}{\rho'} \overset{\Delta}{=} C_L \frac{\langle 1-x\rangle^2}{\rho_L\langle 1-\alpha\rangle} + C_G \frac{\langle x\rangle^2}{\rho_G\langle\alpha\rangle}. \qquad\qquad [15]$$

The parameter ρ' is the generalization of Meyer's (1960) "momentum density", which is not really a density in the physical sense. Its definition, however, makes it possible to write the two-phase momentum equation in the same form as the homogeneous-flow or the single-phase-flow momentum equations. For $C_L = C_G = 1$ [14] becomes the one-dimensional two-phase momentum equation originally derived by Meyer (1960). This form of the mixture momentum equation is currently widely used; however, other forms are possible. First a form of the mixture momentum equation similar to the one presented earlier by Zuber (1967a), and then a Lagrangian form, popular in the Russian literature, are considered.

Drift-velocity formulations of the two-phase momentum equation

Zuber & Findlay (1965) define the local drift velocity of the gas with respect to the local volumetric flux j as,

$$v_{Gj} \overset{\Delta}{=} u_G - j, \qquad\qquad [16]$$

where,

$$j \overset{\Delta}{=} (1-\alpha)u_L + \alpha u_G. \qquad\qquad [17]$$

Using [2], the definition of phase averages, [17] can be averaged over the cross-section,

$$\langle j\rangle = \langle 1-\alpha\rangle\langle u_L\rangle_L + \langle\alpha\rangle\langle u_G\rangle_G. \qquad\qquad [18]$$

Multiplying [16] by α, integrating over the cross-section, and using [7], [18] and Zuber & Findlay's (1965) definitions,

$$C_0 \overset{\Delta}{=} \frac{\langle\alpha j\rangle}{\langle\alpha\rangle\langle j\rangle},$$

and,

$$V_{Gj} \overset{\Delta}{=} \frac{\langle\alpha v_{Gj}\rangle}{\langle\alpha\rangle} = \langle v_{Gj}\rangle_G,$$

the following identities are obtained,

$$\langle u_L \rangle_L = \frac{G}{\langle \rho \rangle} - \frac{\langle \alpha \rangle}{\langle 1 - \alpha \rangle} \frac{\rho_G}{\langle \rho \rangle} [(C_0 - 1)\langle j \rangle + V_{Gj}],$$ [19]

$$\langle u_G \rangle_G = \frac{G}{\langle \rho \rangle} + \frac{\rho_L}{\langle \rho \rangle} [(C_0 - 1)\langle j \rangle + V_{Gj}],$$ [20]

where, [2] and [16] imply,

$$(C_0 - 1)\langle j \rangle + V_{Gj} = \langle u_G \rangle_G - \langle j \rangle.$$ [21]

If [19] is multiplied by $\langle 1 - \alpha \rangle$ and added to $\langle \alpha \rangle$ times [20] we obtain,

$$\langle j \rangle = \frac{G}{\langle \rho \rangle} + \frac{(\rho_L - \rho_G)\langle \alpha \rangle}{\langle \rho \rangle} [(C_0 - 1)\langle j \rangle + V_{Gj}].$$ [22]

Equation [22] is a fundamental identity relating the velocity of the center of volume $\langle j \rangle$ to the velocity of the center of mass U_m which, as will be shown later, is equal to $G/\langle \rho \rangle$. The last term of [22] represents the mass flux drifting out of a control volume moving with the velocity of the center of mass. The $(C_0 - 1)\langle j \rangle$ component of this term is due to the non-uniformity of the velocity and void fraction profiles, while the V_{Gj} component accounts for the local drift between the phases. An equation similar to [22] was presented earlier by Zuber (1967a). This equation was not averaged across the cross section and thus did not contain the $(C_0 - 1)\langle j \rangle$ term. It is evident, however, that this term becomes dominant at high mass fluxes and must be included in one-dimensional analyses. For convenience define:

$$V_{Gj}' \overset{\Delta}{=} (C_0 - 1)\langle j \rangle + V_{Gj}.$$ [23]

The spatial acceleration term of the momentum conservation equation [11] is rewritten in terms of the velocity of the center of mass and V_{Gj} by introducing the correlation coefficients, C_L and C_G:

$$\frac{\partial}{\partial z} \left\{ A_{x-s} \frac{G}{\langle \rho \rangle} [C_L \rho_L \langle 1 - \alpha \rangle \langle u_L \rangle_L + C_G \rho_G \langle \alpha \rangle \langle u_G \rangle_G] \right\}$$

$$+ \frac{\partial}{\partial z} \left\{ \frac{\langle \alpha \rangle \rho_G \rho_L}{\langle \rho \rangle^2} A_{x-s} V_{Gj}' \left[G(C_G - C_L) + \frac{(\langle \alpha \rangle \rho_G C_L + \langle 1 - \alpha \rangle \rho_L C_G)}{\langle 1 - \alpha \rangle} V_{Gj}' \right] \right\}.$$ [24]

In some instances, it might be reasonable to assume that $C_L = C_G = C_{LG}$ (not necessarily a constant). Then the spatial acceleration term can be simply written as,

$$\frac{\partial}{\partial z} \left[\frac{G^2 C_{LG}}{\langle \rho \rangle} A_{x-s} \right] + \frac{\partial}{\partial z} \left[A_{x-s} \frac{\langle \alpha \rangle \rho_G \rho_L}{\langle 1 - \alpha \rangle \langle \rho \rangle} C_{LG} (V_{Gj}')^2 \right].$$ [25]

Finally, adopting the restrictive assumption that $C_L = C_G = 1$, the form of this term previously presented by Zuber (1967a), but with the corrected value V_{Gj}, is obtained. Hence using the identity,

$$\frac{\rho_L - \langle \rho \rangle}{\langle \rho \rangle - \rho_G} = \frac{\langle \alpha \rangle}{\langle 1 - \alpha \rangle}$$ [26]

the cross-sectionally averaged, one-dimensional, two-phase momentum equation of the mixture

can be rewritten in Eulerian form as,

$$-\frac{\partial p}{\partial z} - g\langle\rho\rangle \sin\theta - \frac{\tau_w P_f}{A_{x-s}} - \frac{1}{A_{x-s}}\frac{\partial}{\partial z}\left\{A_{x-s}\left[\frac{\rho_L - \langle\rho\rangle}{\langle\rho\rangle - \rho_G}\right]\frac{\rho_L \rho_G}{\langle\rho\rangle}(V_{Gj}')^2\right\} = \frac{\partial G}{\partial t} + \frac{1}{A_{x-s}}\frac{\partial}{\partial z}\left[\frac{G^2 A_{x-s}}{\langle\rho\rangle}\right].$$

[27]

Several things should be noted about [27]. First of all, since it was derived directly from [11] with the assumption $C_G = C_L = 1$, it contains no more information about momentum conservation than does [14], when the same assumptions are made. That is, integration and numerical evaluation of [14] and [27] produce the same results. However, [27] has more physical appeal since it is written in terms of the velocity of the center of mass and thus uses the correct two-phase density $\langle\rho\rangle$ in the spatial acceleration term, rather than the defined quantity ρ'.

The spatial drift gradient (i.e. the last term on the L.H.S. of [27]), is due to the relative velocity between the phases, and can be considered as an additional volumetric force in the same sense that the classical Reynolds stress term in single-phase turbulence is considered a force (e.g. see Wallis 1968). Physically it represents the net momentum flux with respect to the center of mass of the flowing two-phase mixture. Thus, the center of mass formulation[27] lends itself to a readily understandable interpretation of the various terms and as already noted by Zuber (1966), its derivation is consistent with classical techniques used in the kinetic theory of gases.

Another observation is that the average drift velocity V_{Gj}' appears explicitly in [27], while the average void fraction $\langle\alpha\rangle$ appears in [14] through the definition of ρ'. Although the void fraction can be obtained if one knows the steady-state drift velocity, these two quantities are not necessarily interchangeable under transient conditions. The present state-of-the-art is to use empirical void-quality correlations derived from steady-state experiments to predict the local void fraction from the local, transient flow quality. Under some circumstances it might be more appropriate or convenient to use correlations for the drift velocity and the coefficient C_0. The absence of direct information on V_{Gj}', however, currently limits the practical applications of this formulation.

Finally, the spatial drift gradient can also be written in terms of the relative average velocity U_r, between the phases,

$$U_r \overset{\Delta}{=} \langle u_G\rangle_G - \langle u_L\rangle_L,$$

[28]

through the use of [26] and the following identity,

$$V_{Gj}' = \langle u_G\rangle_G - \langle j\rangle = \langle 1 - \alpha\rangle U_r,$$

[29]

as,

$$\frac{\partial}{\partial z}\left\{A_{x-s}\left[\frac{\rho_L - \langle\rho\rangle}{\langle\rho\rangle - \rho_G}\right]\frac{\rho_L \rho_G}{\langle\rho\rangle}(V_{Gj}')^2\right\} = \frac{\partial}{\partial z}\left[A_{x-s}\frac{\langle\alpha\rangle\langle 1 - \alpha\rangle\rho_L\rho_G}{\langle\rho\rangle}U_r^2\right].$$

For certain applications, in which the relative velocity is known or specified, this may be a more useful form.

Lagrangian forms of the momentum equation

Consider the Lagrangian form of the one-dimensional two-phase momentum equation. Equation [27] can be recast into a Lagrangian formulation by expanding its right hand side and combining it with the continuity equation [6], written in the form,

$$\frac{D_m}{Dt}(\langle\rho\rangle A_{x-s}) + \langle\rho\rangle A_{x-s}\frac{\partial U_m}{\partial z} = 0,$$

[30]

where,

$$\frac{D_m}{Dt} \triangleq \frac{\partial}{\partial t} + U_m \frac{\partial}{\partial z}, \qquad [31]$$

to obtain,

$$\langle \rho \rangle A_{x-s} \frac{D_m U_m}{Dt} + \frac{\partial}{\partial z}\left\{ A_{x-s}\left[\frac{\rho_L - \langle \rho \rangle}{\langle \rho \rangle - \rho_G}\right]\frac{\rho_L \rho_G}{\langle \rho \rangle}(V'_{Gj})^2\right\} = -A_{x-s}\frac{\partial p}{\partial z} - \tau_w P_f - g A_{x-s}\langle \rho \rangle \sin \theta. \quad [32]$$

This is a form of the momentum equation which has previously been used by Ishii & Zuber (1970) to investigate hydrodynamic stability phenomena.

There is another Lagrangian formulation of the two-phase momentum equation which appears in the literature. For a separated flow model, i.e. for $C_G = C_L = 1$, this formulation can be readily derived from [9] and [10]. The right hand side of the liquid-phase momentum equation [9], combined with $\langle u_L \rangle_L / A_{x-s}$ times the liquid-phase continuity equation [4] yields, after expansion of the derivatives and simplification,

$$\sum F_L = \langle 1 - \alpha \rangle \rho_L \left[\frac{\partial \langle u_L \rangle_L}{\partial t} + \langle u_L \rangle_L \frac{\partial \langle u_L \rangle_L}{\partial z}\right] + (1 - \eta)\frac{\delta w'}{A_{x-s}} U_r, \qquad [33]$$

where $\sum F_L$ denotes the sum of the external forces applied to the liquid phase. The last term in [33] can be regarded as a negative volumetric force due to "vapor thrust". That is, it can be interpreted as a reaction force on the liquid due to evaporation. Only for the special case of $\eta = 1$ does this term vanish.

In a similar fashion, the vapor-phase momentum equation can be recast into a Lagrangian formulation by subtracting $\langle u_G \rangle_G / A_{x-s}$ times the vapor continuity equation [3], from the R.H.S. of the vapor momentum equation, [10]:

$$\sum F_G = \langle \alpha \rangle \rho_G \left[\frac{\partial \langle u_G \rangle_G}{\partial t} + \langle u_G \rangle_G \frac{\partial \langle u_G \rangle_G}{\partial z}\right] + \frac{\eta \delta w'}{A_{x-s}} U_r. \qquad [34]$$

For the case of evaporation, the last term in [34] can be regarded as a negative volumetric force due to the slower moving evaporated liquid retarding the vapor phase by momentum transfer.

We are now in a position to write the momentum equations for the liquid and the gaseous phases in terms of the material derivatives of each phase,

$$\frac{D_L}{Dt} \triangleq \frac{\partial}{\partial t} + \langle u_L \rangle_L \frac{\partial}{\partial z}, \qquad [35]$$

$$\frac{D_G}{Dt} \triangleq \frac{\partial}{\partial t} + \langle u_G \rangle_G \frac{\partial}{\partial z}. \qquad [36]$$

Combining [9] and [33], we obtain for the liquid phase,

$$-\langle 1 - \alpha \rangle \frac{\partial p}{\partial z} - g \rho_L \langle 1 - \alpha \rangle \sin \theta - \frac{\tau_w P_f}{A_{x-s}} + \frac{\tau_i P_i}{A_{x-s}} - (1 - \eta)\frac{\delta w'}{A_{x-s}} U_r = \langle 1 - \alpha \rangle \rho_L \frac{D_L \langle u_L \rangle}{Dt}. \qquad [37]$$

Similarly, for the vapor phase, [10] and [34] yield,

$$-\langle \alpha \rangle \frac{\partial p}{\partial z} - g \rho_G \langle \alpha \rangle \sin \theta - \frac{\tau_i P_i}{A_{x-s}} - \frac{\eta \delta w'}{A_{x-s}} U_r = \langle \alpha \rangle \rho_G \frac{D_G \langle u_G \rangle_G}{Dt}. \qquad [38]$$

Equations [37] and [38] can be added to obtain the momentum equation for the two-phase

mixture,

$$-\frac{\partial p}{\partial z} - g\langle\rho\rangle \sin\theta - \frac{\tau_w P_f}{A_{x-s}} - \frac{\delta w'}{A_{x-s}} U_r = \langle 1-\alpha\rangle\rho_L \frac{D_L\langle u_L\rangle_L}{Dt} + \langle\alpha\rangle\rho_G \frac{D_G\langle u_G\rangle_G}{Dt}. \qquad [39]$$

This is a form of the two-phase momentum equation which has been popular in the European and Russian literature. The last term on the left hand side of [39] is the net volumetric force due to evaporation and is frequently referred to as the "Mescherskiy force" in the Russian literature (Kutateladze & Styrikovich 1960). Comparing [32] with [39], it can be seen that a fundamental relationship between the acceleration of the center of mass of the system and the acceleration of the individual phases is given by,

$$\langle\rho\rangle\frac{D_m U_m}{Dt} = \langle 1-\alpha\rangle\rho_L \frac{D_L\langle u_L\rangle_L}{Dt} + \langle\alpha\rangle\rho_G \frac{D_G\langle u_G\rangle_G}{Dt}$$

$$+ \frac{\delta w'}{A_{x-s}} U_r - \frac{1}{A_{x-s}}\frac{\partial}{\partial z}\left\{A_{x-s}\left[\frac{\rho_L-\langle\rho\rangle}{\langle\rho\rangle-\rho_G}\right]\frac{\rho_L\rho_G}{\langle\rho\rangle}(V'_{Gj})^2\right\}. \qquad [40]$$

5. THE ENERGY EQUATIONS

We now turn our attention to the phasic energy equations. In figure 1 the control volume boundary is presumed to be located just inside the liquid phase† (i.e. an infinitesimal distance from the vapor–liquid interface, on the liquid side). For this control volume, conservation of energy for the liquid phase implies,

$$\frac{\partial}{\partial z}\{\rho_L\langle e_L u_L(1-\alpha)\rangle A_{x-s}\}\Delta z + \delta w' e_i \Delta z$$

$$+ p_i\frac{\partial\langle 1-\alpha\rangle}{\partial t} A_{x-s}\Delta z - \zeta\tau_i P_i U_i \Delta z - q_i'' P_i \Delta z - q_L'' P_{h_L}\Delta z$$

$$- q_L''' A_{x-s}\langle 1-\alpha\rangle\Delta z + \frac{\partial}{\partial t}\{\rho_L\langle(e_L - p/\rho_L)(1-\alpha)\rangle A_{x-s}\Delta z\} = 0. \qquad [41]$$

The local values of e_L and e_G are defined by,

$$e_k \stackrel{\Delta}{=} h_k + \frac{u_k^2}{2} + gz\sin\theta, \qquad k = L \text{ or } G, \qquad [42]$$

where h_k denotes the local enthalpy of phase k.

The first term in [41] represents the gradient of convective energy transfer. The second term is the energy transfer due to change of phase, where the specific interfacial energy is given by,

$$e_i \stackrel{\Delta}{=} h_{L_i} + \frac{U_i^2}{2} + gz\sin\theta,$$

where U_i has the same definition as in the momentum equation.

The third term in [41] is the "pdV" reversible work due to changes in control volume size. The fourth term is the fraction of interfacial energy dissipation which crosses the boundary of the control volume. The interfacial dissipation parameter ζ is defined as the net fraction of the energy dissipated at the interface which is transferred across the control volume boundary.

The fifth and sixth terms are the heat-flux terms due to the transfer of sensible and latent heat

---

†It should be noted that one can also use control volumes that do not include the interface and account for the interfacial transfers by "jump conditions". This approach has been taken by Bouré & Réocreux (1972), Delhaye (1974), Bouré (1975), and Ishii (1975).

to the liquid phase at the vapor–liquid interface and the heated wall, respectively. The volumetric heat generation in the liquid phase is given by q'''_L. The last term is the energy rate-of-storage term in the liquid control volume. It should be noted that the interfacial exchange terms would have been different if the control volume for the liquid phase extended beyond the interface, instead of being just within it.

The energy equation for the gas phase can be derived in a similar manner:

$$\frac{\partial}{\partial z}\{\rho_G\langle e_G u_G \alpha\rangle A_{x-s}\}\Delta z + p_i\frac{\partial\langle\alpha\rangle}{\partial t}A_{x-s}\Delta z$$

$$+ q''_i P_i \Delta z + \zeta\tau_i P_i U_i \Delta z - \delta w' e_i \Delta z - q'''_G A_{x-s}\langle\alpha\rangle\Delta z$$

$$- q''_G P_{h_G}\Delta z + \frac{\partial}{\partial t}\{\rho_G\langle(e_G - p/\rho_G)\alpha\rangle A_{x-s}\Delta z\} = 0. \qquad [43]$$

For practical applications, one normally assumes that the interfacial pressure is equal to the phase static pressure (i.e. $p_i = p_G = p_L$).

Using the definitions,

$$q''P_h \overset{\Delta}{=} q''_L P_{h_L} + q''_G P_{h_G},$$

$$q''' \overset{\Delta}{=} q'''_L\langle 1 - \alpha\rangle + q'''_G\langle\alpha\rangle,$$

[41] and [43] can be added to obtain the energy equation of the two-phase mixture:

$$\frac{\partial}{\partial t}\left[\rho_L < \left(e_L - \frac{p}{\rho_L}\right)(1-\alpha) > A_{x-s} + \rho_G < \left(e_G - \frac{p}{\rho_G}\right)\alpha > A_{x-s}\right]$$

$$+ \frac{\partial}{\partial z}[\rho_L\langle e_L u_L (1-\alpha)\rangle A_{x-s} + \rho_G\langle e_G u_G \alpha\rangle A_{x-s}] = q''P_h + q'''A_{x-s}. \qquad [44]$$

It is necessary at this point to again introduce correlation coefficients in order to separate the products $h_L u_L(1-\alpha)$, $u_L(u_L^2/2)(1-\alpha)$ etc., that appear within the angle brackets of [44]. Defining,

$$C'_L \overset{\Delta}{=} \frac{\langle h_L u_L(1-\alpha)\rangle}{\langle h_L\rangle_L\langle u_L(1-\alpha)\rangle} = \frac{\langle h_L u_L(1-\alpha)\rangle}{\langle u_L\rangle_L\langle h_L(1-\alpha)\rangle} = \frac{\langle h_L u_L(1-\alpha)\rangle}{\langle h_L\rangle_L\langle u_L\rangle_L\langle 1-\alpha\rangle},$$

$$\qquad [45]$$

$$C'_G \overset{\Delta}{=} \frac{\langle h_G u_G \alpha\rangle}{\langle h_G\rangle_G\langle u_G \alpha\rangle} = \frac{\langle h_G u_G \alpha\rangle}{\langle u_G\rangle_G\langle h_G \alpha\rangle} = \frac{\langle h_G u_G \alpha\rangle}{\langle h_G\rangle_G\langle u_G\rangle_G\langle\alpha\rangle},$$

and

$$C''_L \overset{\Delta}{=} \frac{\langle u_L(u_L^2/2)(1-\alpha)\rangle}{(1/2)\langle u_L\rangle_L^2\langle u_L(1-\alpha)\rangle} = \frac{\langle u_L(u_L^2/2)(1-\alpha)\rangle}{(1/2)\langle u_L\rangle_L^3\langle 1-\alpha\rangle},$$

$$\qquad [46]$$

$$C''_G \overset{\Delta}{=} \frac{\langle u_G(u_G^2/2)\alpha\rangle}{(1/2)\langle u_G\rangle_G^2\langle u_G \alpha\rangle} = \frac{\langle u_G(u_G^2/2)\alpha\rangle}{(1/2)\langle u_G\rangle_G^3\langle\alpha\rangle}.$$

Also defining for convenience the "total energy" correlation coefficients,

$$K_L \overset{\Delta}{=} \frac{\langle e_L u_L(1-\alpha)\rangle}{\langle e_L\rangle_L\langle u_L\rangle_L\langle 1-\alpha\rangle} = \frac{[C'_L\langle h_L\rangle_L + C''_L(1/2)\langle u_L\rangle_L^2 + gz\sin\theta]}{[\langle h_L\rangle_L + \langle u_L^2/2\rangle_L + gz\sin\theta]},$$

$$\qquad [47]$$

$$K_G \overset{\Delta}{=} \frac{\langle e_G u_G \alpha\rangle}{\langle e_G\rangle_G\langle u_G\rangle_G\langle\alpha\rangle} = \frac{[C'_G\langle h_G\rangle_G + C''_G(1/2)\langle u_G\rangle_G^2 + gz\sin\theta]}{[\langle h_G\rangle_G + \langle u_G^2/2\rangle_G + gz\sin\theta]},$$

where the cross-sectional averages for each phase are obtained using [2],

$$\langle e_k \rangle_k = \langle h_k \rangle_k + \langle u_k^2/2 \rangle_k + gz \sin \theta, \qquad \text{for } k = L \text{ or } G, \qquad [48]$$

the energy equation of the mixture [44], can then be written as,

$$A_{x-s} \frac{\partial}{\partial t} \left[\rho_L \langle 1 - \alpha \rangle \left(\langle e_L \rangle_L - \frac{p}{\rho_L} \right) + \rho_G \langle \alpha \rangle \left(\langle e_G \rangle_G - \frac{p}{\rho_G} \right) \right]$$
$$+ \frac{\partial}{\partial z} [K_L w_L \langle e_L \rangle_L + K_G w_G \langle e_G \rangle_G] = q'' P_h + q''' A_{x-s} \qquad [49]$$

where w_L and w_G are the mass flow rates of the liquid and gas phases, respectively. Using [13] this equation can be rewritten in terms of quality and total mass flux as,

$$\frac{\partial}{\partial t} (\langle \rho \rangle \bar{h}) A_{x-s} + \frac{\partial}{\partial z} (G A_{x-s} h') = q'' P_h + q''' A_{x-s}$$
$$- \frac{\partial}{\partial z} \left[\frac{G^3 A_{x-s}}{2(\rho''')^2} \right] - g G A_{x-s} \sin \theta - \frac{\partial}{\partial t} \left[\frac{G^2}{2\rho'} \right] + \frac{\partial p}{\partial t} A_{x-s} \qquad [50]$$

where the two-phase mixture continuity equation, [6], has been used to simplify the potential energy term, and we have defined,

$$\bar{h} \overset{\Delta}{=} \frac{1}{\langle \rho \rangle} [\rho_L \langle 1 - \alpha \rangle \langle h_L \rangle_L + \rho_G \langle \alpha \rangle \langle h_G \rangle_G] \qquad [51]$$

$$h' \overset{\Delta}{=} C'_L \langle 1 - x \rangle \langle h_L \rangle_L + C'_G \langle x \rangle \langle h_G \rangle_G \qquad [52]$$

$$\frac{1}{(\rho''')^2} \overset{\Delta}{=} C''_L \frac{\langle 1 - x \rangle^3}{\rho_L^2 \langle 1 - \alpha \rangle^2} + C''_G \frac{\langle x \rangle^3}{\rho_G^2 \langle \alpha \rangle^2} \qquad [53]$$

and ρ' has been defined by [15] already.

A remark about the correlation coefficients appearing in the energy conservation equation is in order. First, for many cases of interest in adiabatic flow or bulk boiling, the enthalpies of each phase are uniform across the cross-section. In this case C'_L and C'_G are equal to unity. It is only in the presence of thermodynamic non-equilibrium that C'_L and C'_G are different from unity. Second, changes in the terms $\langle u_L^3 (1 - \alpha) \rangle$ and $\langle u_G^3 \alpha \rangle$, representing the kinetic energy of each phase, are often much smaller than the corresponding changes in the enthalpy terms $\langle h_L u_L (1 - \alpha) \rangle$ and $\langle h_G u_G \alpha \rangle$. Unfortunately, little experimental information is currently available to calculate the values of the correlation coefficients. Thus it is often either acceptable or necessary to assume that the values of all the correlation coefficients appearing in the energy equation are equal to unity. In this case h' and ρ''' take the forms usually found in the literature (Meyer 1960).

For the special case of a saturated two-phase mixture, in which the properties are only a function of h' it may be more convenient to expand the left hand side of [50] and combine it with [6] to yield an equation similar in form, but more general, than the one derived by Meyer (1960).

$$\rho'' \frac{\partial h'}{\partial t} + G \frac{\partial h'}{\partial z} = \frac{q'' P_h}{A_{x-s}} + q''' - \frac{1}{A_{x-s}} \frac{\partial}{\partial z} \left[\frac{G^3 A_{x-s}}{2(\rho''')^2} \right] - g G \sin \theta - \frac{1}{A_{x-s}} \frac{\partial}{\partial t} \left[\frac{G^2}{2\rho'} \right] + \frac{\partial p}{\partial t}, \qquad [54]$$

where

$$\rho'' \overset{\Delta}{=} \left[\langle \rho \rangle \frac{d\bar{h}}{dh'} + (\bar{h} - h') \frac{d\langle \rho \rangle}{dh'} \right]. \qquad [55]$$

The explicit functional form of ρ'' for various special cases of interest has been given by Meyer (1960). Equation [54] has been widely used; however, one is required to deal with several pseudo-densities (ρ', ρ'', ρ'''). Moreover, [54] cannot be cast into a true Lagrangian formulation and thus it has a limited physical interpretation.

Lagrangian forms of the energy equation

Zuber (1967a) has introduced an alternate form of the energy equation which more readily lends itself to phenomenological interpretation. This equation is now derived, without making any assumptions regarding the values of the correlation coefficients. Equation [43] can be rewritten as,

$$A_{x-s}\frac{\partial}{\partial t}[\langle\rho\rangle\bar{e}]+\frac{\partial}{\partial z}\{A_{x-s}[\rho_L\langle e_L u_L(1-\alpha)\rangle+\rho_G\langle e_G u_G\alpha\rangle]\} = A_{x-s}\frac{\partial p}{\partial t}+P_h q''+A_{x-s}q''' \qquad [56]$$

where

$$\bar{e} \triangleq \frac{\rho_L\langle e_L\rangle_L\langle 1-\alpha\rangle+\rho_G\langle e_G\rangle_G\langle\alpha\rangle}{\langle\rho\rangle}. \qquad [57]$$

Using the correlation coefficients, K_L and K_G, the identities [19] and [20], the definition of the total derivative, [31], and the mass continuity equation [6] to eliminate some terms, [56] can be rewritten in a Lagrangian form as,

$$\langle\rho\rangle A_{x-s}\frac{D_m\bar{e}}{Dt}-\frac{\partial}{\partial z}\left[A_{x-s}G\delta e^* - A_{x-s}e_{LG}^*\frac{\langle\alpha\rangle}{\langle\rho\rangle}\rho_L\rho_G V_{Gj}'\right] = A_{x-s}\frac{\partial p}{\partial t}+P_h q''+A_{x-s}q''', \qquad [58]$$

where

$$\delta e^* \triangleq \frac{(1-K_L)\rho_L\langle e_L\rangle_L\langle 1-\alpha\rangle+(1-K_G)\rho_G\langle e_G\rangle_G\langle\alpha\rangle}{\langle\rho\rangle}, \qquad [59]$$

and

$$e_{LG}^* \triangleq K_G\langle e_G\rangle_G - K_L\langle e_L\rangle_L. \qquad [60]$$

The term in brackets on the L.H.S. of [58] represents the drift of energy through the center-of-mass plane. The first term, $A_{x-s}G\delta e^*$, is due to the non-uniformity of the radial velocity and energy distributions, while the second term accounts for the unequal velocities of the two phases as evidenced by the presence of V_{Gj}'.

For $K_L = K_G = 1$ one obtains,

$$\langle\rho\rangle A_{x-s}\frac{D_m\bar{e}}{Dt}+\frac{\partial}{\partial z}\left[A_{x-s}\frac{\langle\alpha\rangle\rho_L\rho_G}{\langle\rho\rangle}V_{Gj}'e_{LG}\right] = A_{x-s}\frac{\partial p}{\partial t}+P_h q''+A_{x-s}q''', \qquad [61]$$

where e_{LG}^* has become the familiar,

$$e_{LG} \triangleq \langle e_G\rangle_G - \langle e_L\rangle_L.$$

With the exception that V_{Gj}' is used rather than V_{Gj}, [61] is the form of the two-phase energy conservation equation presented earlier by Zuber (1967a).

Making the assumption of separated two-phase flow again, i.e. assuming that $K_L = K_G = 1$, another Lagrangian form of the two-phase energy equation can be obtained. By combining [3] and [4] with [49],

$$\rho_L\langle 1-\alpha\rangle\frac{D_L\langle e_L\rangle_L}{Dt}+\rho_G\langle\alpha\rangle\frac{D_G\langle e_G\rangle_G}{Dt} = \frac{q''P_h}{A_{x-s}}+q'''+\frac{\partial p}{\partial t}-\frac{\delta w'}{A_{x-s}}e_{LG}. \qquad [62]$$

This form is compatible with the form of the two-phase momentum equation given by [39]. The last term on the right hand side of [62] represents the volumetric rate of energy transfer due to the evaporation process. Equation [62] is frequently used for evaluating $\delta w'$. For the special case of saturated equilibrium, constant pressure, and negligible internal heat generation (q'''), kinetic and potential energy, [62] yields [5].

It should be obvious that the same information on energy conservation is contained in all the separated-flow model energy equations, namely in [62] and the simplified forms of [49], [50] and [61]. Furthermore, numerous other forms are derivable from [49].

### 6. NEW LAGRANGIAN FORMS OF THE CONSERVATION EQUATIONS

For certain types of two-phase flow problems Lagrangian formulations of the conservation equations result in simple, exact solutions. For example, Gonzalez-Santalo & Lahey (1973) and Shiralkar et al. (1973) respectively, have used such formulations for homogeneous and slip flow, to study transients in boiling water nuclear reactors. The key to such analyses is the proper formulation of the conservation equations. Alternative, potentially useful, Lagrangian forms of the conservation equations are now derived.

Consider the velocities shown schematically in figures 2a–2c. It can be shown that U_m, defined earlier as $G/\langle\rho\rangle$, is the velocity of the center-of-mass of the two-phase system; i.e. it is the velocity of propagation of the plane through which no net mass flux passes. Indeed, equating the mass flux terms in figure 2a,
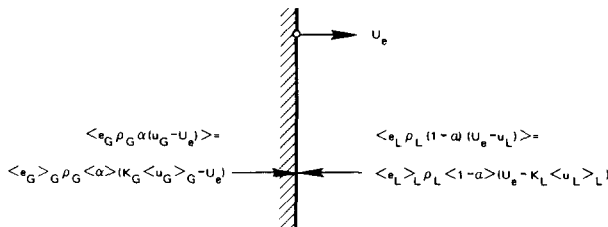
$$\rho_G \langle\alpha\rangle[\langle u_G\rangle_G - U_m] = \rho_L \langle 1 - \alpha\rangle[U_m - \langle u_L\rangle_L] .$$
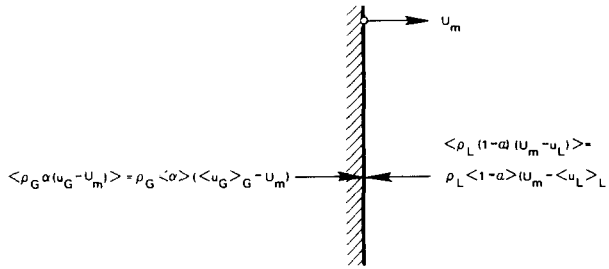
a. THE VELOCITY OF THE CENTER-OF-MASS, U_m
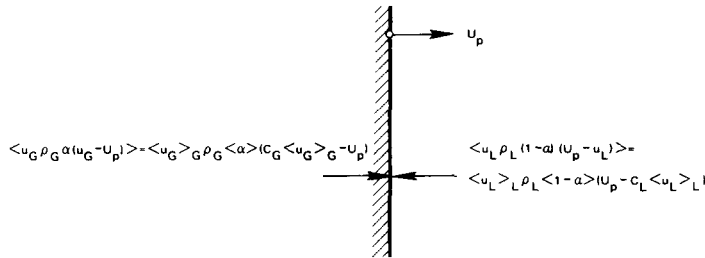
b. THE VELOCITY OF THE CENTER -OF-MOMENTUM, U_p

c. THE VELOCITY OF THE CENTER-OF-ENERGY, U_e

Figure 2. Two-phase velocities.

Solving for U_m we obtain,

$$U_m \triangleq \frac{\rho_L \langle u_L \rangle_L \langle 1 - \alpha \rangle + \rho_G \langle u_G \rangle_G \langle \alpha \rangle}{\langle 1 - \alpha \rangle \rho_L + \langle \alpha \rangle \rho_G} = \frac{G}{\langle \rho \rangle}.$$ [63]

It has already been shown that use of a material derivative defined using the velocity of the center-of-mass results in relatively simple forms of the conservation equations ([30], [32], and [61]) clearly showing the effects of the drift fluxes. It is possible to extend this idea and define velocities of the center-of-volume, the center-of-momentum, and the center-of-energy of the mixture. The velocity of the center-of-volume of the mixture is simply the volumetric flux, $\langle j \rangle$. Expressions for the velocities of the center-of-momentum and center-of-energy are now derived.

Equating the momentum flux terms in figure 2b to define the plane through which zero net momentum flux passes,

$$\langle u_G \rangle_G \rho_G \langle \alpha \rangle (C_G \langle u_G \rangle_G - U_p) = \langle u_L \rangle_L \rho_L \langle 1 - \alpha \rangle (U_p - C_L \langle u_L \rangle_L)$$

we obtain using [7], [13] and [15],

$$U_p \triangleq \frac{C_L \rho_L \langle u_L \rangle_L^2 \langle 1 - \alpha \rangle + C_G \rho_G \langle u_G \rangle_G^2 \langle \alpha \rangle}{\rho_L \langle u_L \rangle_L \langle 1 - \alpha \rangle + \rho_G \langle u_G \rangle_G \langle \alpha \rangle} = \frac{G}{\rho'}.$$ [64]

Finally, defining the plane through which no net energy flux passes, by equating the energy flux terms in figure 2c,

$$\langle e_G \rangle_G \rho_G \langle \alpha \rangle (K_G \langle u_G \rangle_G - U_e) = \langle e_L \rangle_L \rho_L \langle 1 - \alpha \rangle (U_e - K_L \langle u_L \rangle_L),$$

we obtain,

$$U_e \triangleq \frac{K_L \rho_L \langle e_L \rangle_L \langle 1 - \alpha \rangle \langle u_L \rangle_L + K_G \rho_G \langle e_G \rangle_G \langle \alpha \rangle \langle u_G \rangle_G}{\rho_L \langle 1 - \alpha \rangle \langle e_L \rangle_L + \rho_G \langle \alpha \rangle \langle e_G \rangle_G} = \frac{Ge'}{\bar{e} \langle \rho \rangle}$$ [65]

where use of [13] and [57] has been made and

$$e' \triangleq K_L \langle 1 - x \rangle \langle e_L \rangle_L + K_G \langle x \rangle \langle e_G \rangle_G = K_L \langle e_L \rangle_L + \langle x \rangle e_{LG}^*.$$ [66]

The continuity equation in terms of the velocity of the center-of-mass has already been given as,

$$\frac{D_m}{Dt} (\langle \rho \rangle A_{x-s}) + \langle \rho \rangle A_{x-s} \frac{\partial U_m}{\partial z} = 0.$$ [30]

The momentum equation can be written in terms of the velocity of the center-of-momentum by expanding the R.H.S. of [14] and combining it with [64] to yield,

$$-\frac{\partial p}{\partial z} - g \langle \rho \rangle \sin \theta - \frac{\tau_w P_f}{A_{x-s}} = \frac{1}{A_{x-s}} \frac{D_p (GA_{x-s})}{Dt} + G \frac{\partial U_p}{\partial z}$$ [67]

where,

$$\frac{D_p}{Dt} \triangleq \frac{\partial}{\partial t} + U_p \frac{\partial}{\partial z}.$$

Finally, the energy equation can be written in terms of the velocity of the center of energy by

introducing [13] into [49] and regrouping the terms using [57] and [65] to yield,

$$\frac{1}{A_{x-s}}\frac{D_e}{Dt}[\langle\rho\rangle\bar{e}A_{x-s}]+\langle\rho\rangle\bar{e}\frac{\partial U_e}{\partial z}=\frac{q''P_h}{A_{x-s}}+q'''+\frac{\partial p}{\partial t} \qquad [68]$$

where,

$$\frac{D_e}{Dt}\triangleq\frac{\partial}{\partial t}+U_e\frac{\partial}{\partial z}\,.$$

Clearly the conservation equations given by [30], [67], and [68] are quite compact and phenomenologically appealing. Nevertheless, one must deal with a different velocity in each equation. Since there are no two-phase dynamics problems that can be solved using only one of the three conservation equations, these formulations yield no fundamental simplification; however, they may be convenient for certain applications. Other forms are also possible.

For instance one can define a momentum-mass drift velocity,

$$V_{pm}\triangleq U_p-U_m$$

and an energy–mass drift velocity,

$$V_{em}\triangleq U_e-U_m\,.$$

These drift velocities have an interesting physical meaning, although their usefulness is currently limited by the fact that empirical correlations for them have not yet been developed. Nevertheless, the conservation equations can be rewritten in terms of U_m, V_{pm} and V_{em} as,

$$\frac{D_m}{Dt}(\langle\rho\rangle A_{x-s})+\langle\rho\rangle A_{x-s}\frac{\partial U_m}{\partial z}=0\,. \qquad [30]$$

$$\langle\rho\rangle A_{x-s}\frac{D_m U_m}{Dt}+\frac{\partial}{\partial z}(GA_{x-s}V_{pm})=-A_{x-s}\frac{\partial p}{\partial z}-\tau_w P_f-g\langle\rho\rangle A_{x-s}\sin\theta. \qquad [69]$$

$$\langle\rho\rangle A_{x-s}\frac{D_m\bar{e}}{Dt}+\frac{\partial}{\partial z}[\bar{e}\langle\rho\rangle A_{x-s}V_{em}]=A_{x-s}\frac{\partial p}{\partial t}+q''P_h+q'''A_{x-s}\,. \qquad [70]$$

The second terms on the L.H.S. of the momentum and energy conservation equations clearly represent the momentum and energy drift fluxes through the center-of-mass plane. For certain cases of interest, over a limited range of variables, the drift flux gradients might be negligible, or analytically tractable. In these cases one would obtain a set of equations that have the simplicity and ease of analytical solution of the homogeneous formulation, and yet retain the more realistic character of the present formulations. Moreover the drift flux terms of [69] and [70] may be used to quantify the approximations involved when homogeneous-flow models are used.

7. CONCLUSIONS

The conservation equations of two-phase flow can be manipulated and recast into many different forms. We have not tabulated all the possible forms, but rather, discussed and related several popular forms which frequently appear in the literature of two-phase flow. The choice of a form depends on the particular problem to be solved and, in many cases, may be largely a matter of convenience or taste.

Spatial correlation coefficients accounting for the radial distribution of void fraction, velocity and enthalpy of each phase have been systematically introduced into the conservation equations. The generalized formulations given in this paper are an improvement over the classical

separated-flow model equations, and are recommended for the analysis of two-phase flow phenomena.

Acknowledgement—The authors gratefully acknowledge the helpful discussions and comments from Drs. J. Bouré and J. M. Delhaye during the preparation of this paper.

REFERENCES

BIJWAARD, G., STAUB, F. & ZUBER, N. A program of two-phase flow investigation. 7th quarterly report—October–December 1964. GEAP-4778.

BOURÉ, J. A. & RÉOCREUX, M. L. 1972 General equations of two-phase flows. Applications to critical flows and to non-steady flows. Fourth All-Union Heat and Mass Transfer Conference, Minsk.

BOURÉ, J. A. 1975 On a unified presentation of the non-equilibrium two-phase flow models. *ASME Symp. Vol. on Non-Equilibrium Two-Phase Flows.*

DELHAYE, J. M. 1968 Equations fondamentales des écoulements diphasiques, Centre d'Etudes Nucléaires de Grenoble Report CEA-R-3429 (1) and (2).

DELHAYE, J. M. 1974 Jump conditions and entropy sources in two-phase systems. Local instant formulation. *Int. J. Multiphase Flow* **1**, 395–409.

GONZALEZ-SANTALO, J. M. & LAHEY, R. T., JR. 1973 An exact solution of flow decay transients in two-phase systems by the method of characteristics. *J. Heat Transfer* **95C**, 470–476.

HANCOX, W. T. & NICOLL, W. B. 1971 Prediction of time-dependent two-phase water flows. In *Progress in Heat and Mass Transfer* (Edited by HETSRONI, G., SIDEMAN, S. & HARTNETT, J. P.), Vol. 6, 119–135. Pergamon Press, Oxford.

ISHII, M. & ZUBER, N. 1970 Thermally induced flow instabilities in two-phase mixtures. *Heat Transfer* 1970, Proceedings of the fourth International Heat Transfer Conference, Paris, Vol. V, Paper B5.11.

ISHII, M. 1975 *Thermo-Fluid Dynamic Theory of Two-Phase Flow.* Eyrolles, Paris.

KOCAMUSTAFAOGULLARI, G. 1971 Thermo-fluid dynamics of separated two-phase flow. Ph.D. Thesis, Georgia Institute of Technology.

KUTATELADZE, S. S. & STYRIKOVICH, M. A. 1960 *Hydraulics of Gas–Liquid Systems*, p. 24. Wright–Patterson Air Force Base translation.

LAHEY, R. T., JR. 1974 Two-phase flow in boiling water nuclear reactors. NEDO-13388.

MEYER, J. E. 1960 Conservation laws in one-dimensional hydrodynamics. *Bettis Tech. Rev.* WAPD-BT-20, 61–72.

RÉOCREUX, M. 1973 Débits critiques. Centre d'Etudes Nucléaires de Grenoble Report CEA-N-1639.

SHIRALKAR, B. S., SCHNEBLY, L. E. & LAHEY, R. T., JR. 1973 Variation of the vapor volumetric fraction during flow and power transients. *Nucl. Engng Design* **25**, 350–368.

VERNIER, P. & DELHAYE, J. M. 1968 General two-phase flow equations applied to the thermodynamics of boiling nuclear reactors. *Energie Primaire* **4**, 5–36.

WALLIS, G. B. 1968 Use of the Reynolds flux concept for analysing one-dimensional two-phase flow, Part II. Application to two-phase flow. *Int. J. Heat Mass Transfer* **11**, 459–472.

WALLIS, G. B. 1969 *One-Dimensional Two-phase Flow.* McGraw-Hill, New York.

WUNDT, H. 1967 Basic relationships in *n*-component diabatic flow, EUR 3459.e; also: 1967 On the structure of consistent equation sets to describe multiphase flow. *Proc. Symp. on Two-Phase Flow Dynamics.* Euratom—The Technological University of Eindhoven.

ZUBER, N. & FINDLAY, J. A. 1965 Average volumetric concentration in two-phase flow systems. *J. Heat Transfer* **87C**, 453–468.

ZUBER, N. & STAUB, F. W. 1966 The propagation and the wave form of the vapor volumetric concentration in boiling, forced convection system under oscillatory conditions. *Int. J. Heat Mass Transfer* **9**, 871–895.

ZUBER, N. & STAUB, F. W. 1967 An analytical investigation of the transient response of the volumetric concentration in a boiling forced-flow system. *Nucl. Sci. Engng* **30**, 268–278.

ZUBER, N. 1967a Flow excursions and oscillations in boiling, two-phase flows with heat addition. *Proc. Symp. on Two-phase Flow Dynamics.* Euratom—The Technological University of Eindhoven.

ZUBER, N. 1967b Conservation laws for two-phase flow with a change of phase. *Int. J. Heat Mass Transfer* **10**, 1637–1642.

ZUBER, N. & DOUGHERTY, D. E. 1967 Liquid metals challenge to the traditional method of two-phase flow investigations. *Proc. Symp. on Two-Phase Flow Dynamics.* Euratom—The Technological University of Eindhoven.